The Singularity Theory Problem List

A list of problems related to Singularity Theory by and for its community

This is a (very) preliminary version, visit https://rgimenezconejero.github.io/list.html

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Chapter 1

Singular varieties

1.1 Classification of algebraic curves and surfaces

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Classification of algebraic curves and surfaces is the big mathematical problem over the years. Here are two problems to study, the first one is related to the classification of algebraic surfaces, and the second one relates to the classification of algebraic curves.

Problem 1.1. We take a deformation of an algebraic surface to a union of planes and project it onto the projective plane. We compute the fundamental group of the complement of the branch curve in the projective plane, and also other related fundamental groups. Those fundamental groups are invariants of the surfaces and help us to classify the surfaces in the moduli space. Surfaces in the same connected component in the moduli space will have the same related fundamental group.

Moreover, we have singularities of many interesting types in those deformations, it is interesting and challenging to find their regenerations and determine fundamental groups. The study of non-planar deformations can supply appearance of special and complicated singularities.

Studying non-planar deformations of degree 8 and higher than this, check the fundamental groups of the complements of the branch curves.

The study of deformations to unions of planes involves also the study of Zappatic singularities. But if we study much more complicated singularities than Zappatic ones, or Zappatic singularities with higher multiplicity, then additional tools like homotopy techniques, special softwares, or specific results in the theory of singularities could be considered. In particular, non-planar deformations (degenerations) are intriguing because these might reveal more intricate or exotic singularities.

This study can direct us to study Galois covers of algebraic surfaces, because the fundamental group of the Galois cover of a surface is also an invariant of the classification. It is a special quotient of the fundamental group of the complement of the branch curve of the surface. It is interesting to study this group because there is a use of Coxeter or Artin groups to determine fundamental groups.

One of the promising goals is the work on how these fundamental groups change under deformations and what they tell us about moduli spaces of surfaces.

References: [Amr25, AGR+24, AGM23, AGM24a, AGM24b, ALV12, Lib85, Lib07, Lib14, CCFM08, CCFM07, CLM93]

Key words: Deformation of surfaces, regeneration of singularities, Zariski pairs, fundamental groups

Problem 1.2. Concerning algebraic curves, we can study line arrangements and conic line arrangements, and find Zariski pairs. Much of the work is focused on conic-line arrangements of degree equal or higher than 7.

Also deformations of plane curves can be very interesting. So we take the above arrangements and we study the possible deformations and the differences in the groups.

The study of algebraic curve classification has a connection with line arrangements, stick curves and Zariski pairs; this could yield additional insights into how the topology of the curves relates to their algebraic classification.

Exploring the fundamental groups of the complements of algebraic curves through deformation theory could provide further connections to other areas, such as the classification of singularities or the study of moduli spaces for algebraic curves.

References: [ASS+23, ABS+23, ABST20, BT20, ABCAMM19, ABCT08]

Chapter 2

Singular mappings

2.1 Algebra-computing spectral sequences

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In the theory of singularities of maps, also called *Thom-Mather theory*, one of the commonly studied topics is the topology of the discriminants of maps and how they change after small perturbations. In close connection with this, the algebraic properties of the singularities and their deformations are often expressed in terms of certain *codimensions*, *versal unfoldings*, *bifurcations sets*, etc. Some modern references for this are [MNB20] and [**?**, Chapter 2].

In the case that the dimension of the target space of a map germ, say p, is greater than the dimension of the source, say n, the discriminant coincides with the image. There is a tool to compute the homology of the image of a map that works in great generality, called the *Image-Computing Spectral Sequence* (ICSS). However, there is no such tool to help us to compute the algebra of the singularity of a map, something similar to an *Algebra-Computing Spectral Sequence* (ACSS).

The ICSS of a map $f : N \to P$ uses the *homological properties* of the multiple point spaces of f $D^k(f)$ to compute (to some extent) the homology of $f(N)$ and, for this reason, one hopes that there is an ACSS that uses the *algebraic information* of the multiple point spaces to compute the algebraic invariants we want. More precisely:

Problem 2.1. Find a spectral sequence that, for any map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, computes the $\mathscr{A}_e$-codimension of f using algebraic information of the multiple point spaces $D^k(f)$.

The case of Problem 2.1 for corank one map germs should be simpler, since the multiple point spaces are isolated complete intersection singularities when the germ has finite $\mathscr{A}_e$-codimension. See [MM89, Proposition 2.14].

Problem 2.2. For corank one map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, there is a spectral sequence that uses the Tjurina modules of the multiple point spaces $D^k(f)$ and computes the $\mathscr{A}_e$-codimension of f.

Some recommended references for the ICSS are [GM93, Hou07, CMM22] and [GC21, Chapter 2]. This problem was first stated in the Remark 7.7 of the related work [GC22].

Key words: Singularities of maps, Thom-Mather theory, codimension, spectral sequence, computation

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